A regularized interior-point method for optimization problems with complementarity constraints

François Pacaud

frapac.github.io

CAS, Mines Paris - PSL

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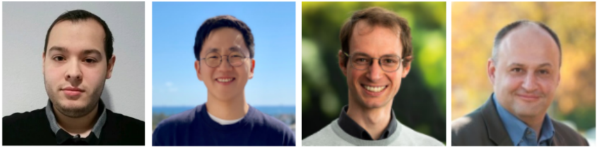
Who are we?

- An international team looking at the future of nonlinear programming



Outline: how to solve complementarity problems with interior-point?

1. We give a brief recall of programs with complementarity constraints (and why they are difficult)
2. We present NCL, a mixed augmented-Lagrangian/interior-point algorithm
3. We test the method on large-scale SC-OPF instances and present recent numerical results

Part I: Complementarity constraints

- What is a program with complementarity constraints?
- How to solve them using interior-point?

Nonlinear program with complementarity constraints

Mathematical program with complementarity constraints (MPCC)

Let $f : \mathbb{R}^n \to \mathbb{R}$, $G : \mathbb{R}^n \to \mathbb{R}^m$ and $H : \mathbb{R}^n \to \mathbb{R}^m$.
The MPCC is here defined as:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \; & f(x) \\ \text{s.t.} \quad & 0 \leq G(x) \perp H(x) \geq 0 \end{aligned}$$

Notation $0 \leq x \perp y \geq 0$ stands for:

$$0 \leq x_i \quad \text{and} \quad 0 \leq y_i \quad \text{and} \quad x_i y_i = 0 \quad \forall i = 1, \cdots, m$$

NLP reformulation

MPCC is equivalent to the NLP:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \; & f(x) \\ \text{s.t.} \quad & G(x) \geq 0\,, \quad H(x) \geq 0 \\ & G_i(x) H_i(x) \leq 0 \quad \forall i = 1, \cdots, m \end{aligned}$$

- The previous NLP is degenerate, in the sense that MFCQ fails at all x

Optimality conditions

- Active sets:

$$I_G = \{i \in \{1, \cdots, m\} \mid G_i(x) = 0\}\ , \quad I_H = \{i \in \{1, \cdots, m\} \mid H_i(x) = 0\}\ .$$

- MPCC Lagrangian:

$$L(x, \lambda_G, \lambda_H) := f(x) - \lambda_G^\top G(x) - \lambda_H^\top H(x)\ .$$

Stationarity conditions

Weak stationary $\nabla_x L(x, \lambda) = 0$ with

$$(\lambda_G)_i = 0 \quad \text{for} \quad i \notin I_G \quad \text{and} \quad (\lambda_H)_i = 0 \quad \text{for} \quad i \notin I_H$$

Clarke stationary: weak stationary and

$$(\lambda_G)_i(\lambda_H)_i \geq 0 \quad \forall i \in I_G \cap I_H$$

Mordukhovich stationary: Clarke stationary and

$$\text{Either}\Big((\lambda_G)_i > 0 \text{ and } (\lambda_H)_i > 0\Big) \text{ or } (\lambda_G)_i(\lambda_H)_i = 0 \qquad \forall i \in I_G \cap I_H$$

Strong stationary: Mordukhovich stationary and

$$(\lambda_G)_i \geq 0 \quad \text{and} \quad (\lambda_H)_i \geq 0 \quad \forall i \in I_G \cap I_H$$

Usual solution methods

- **Relaxation:** For $\tau > 0$, replace complementarity constraints by

$$G_i(x)H_i(x) \leq \tau \;, \quad \forall i = 1, \cdots, m$$

- **Smoothening:** Use a smooth approximation parameterized by $\varepsilon > 0$:

$$\Phi_\varepsilon\Big(G_i(x), H_i(x)\Big) = 0 \quad \forall i = 1, \cdots, m$$

- **Exact penalty:** Move the complementarity constraints in the objective:

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} \; f(x) + \nu G(x)^\top H(x) \\ &\text{s.t.} \quad G(x) \geq 0 \;, \quad H(x) \geq 0 \end{aligned}$$

  and solve resulting problem with interior-point

R. Fletcher and S. Leyffer. "Solving mathematical programs with complementarity constraints as nonlinear programs." Optimization Methods and Software

Part II: NCL

- How to mix interior-point with an Augmented Lagrangian method?
- Is the method more complex than vanilla interior-point?

Augmented Lagrangian method

Nonlinear program

$$\min_{x \in \mathbb{R}^n} f(x) \quad \text{subject to} \quad c(x) = 0\ ,\ x \geq 0$$

For $y_e \in \mathbb{R}^m, \rho > 0$, we define the Augmented Lagrangian subproblem:

$$\min_{x \in \mathbb{R}^n} f(x) + y_e^\top c(x) + \frac{\rho}{2}\|c(x)\|^2 \quad \text{subject to} \quad x \geq 0$$

Introducing the NCL algorithm

NCL reformulates the Auglag's subproblems as constrained optimization problems

At iteration k, the algorithm solves:

$$\min_{x,r} f(x) - (y_k^e)^\top r + \frac{\rho_k}{2}\|r\|^2 \qquad (\mathrm{NCL}_k)$$
$$\text{subject to} \quad c(x) + r = 0\,, \quad x \geq 0$$

- Subproblem (NCL_k) is always feasible, solvable by **IPM**!
- Only the objective changes between two Auglag iterations
- Regularization r stabilizes internal IPM iterations

NCL algorithm $\equiv$ Auglag algorithm

- Solve (NCL_k) to get (x_k, r_k), for a tolerance ω_k
- Update parameters as
  - If $\|c(x_k)\| \leq \eta_k$, set $y_{k+1}^e = y_k^e - \rho_k r_k$
  - Else $\rho_{k+1} = 10 \times \rho_k$.

D. Ma, KL. Judd, D. Orban and MA Saunders. "Stabilized optimization via an NCL algorithm".

Writing the KKT system

Compared to raw IPM, each NCL's subproblem has an additional variable r

KKT system: NCL

NCL search direction is computed by solving the KKT system:

$$\begin{bmatrix} W + \Sigma_x & 0 & J^\top \\ 0 & \rho_k I & I \\ J & I & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta r \\ \Delta y \end{bmatrix} = - \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} \qquad (K_2)$$

Note that the NCL Jacobian is always full row-rank

Condensation strategy

Step 1: removing Δr

Eliminate $\Delta r = (n_2 - \Delta y)/\rho_k$:

$$\begin{bmatrix} W + \Sigma_x & J^\top \\ J & -\frac{1}{\rho_k} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = - \begin{bmatrix} n_1 \\ n_3 - \frac{1}{\rho_k} n_2 \end{bmatrix} \qquad (K_{2r})$$

Step 2: removing Δy

Eliminate $\Delta y = n_2 - \rho_k(n_3 - J\Delta x)$:

$$(W + \Sigma_x + \rho_k J^\top J)\Delta x = J^\top (n_3 + \rho_k r_1) - r_2 \qquad (K_{1s})$$

The original problem is nonconvex, hence:

- K_{2r} is (almost) SQD (LDL)
- K_{1s} is (almost) positive definite (Cholesky)

Safeguarding convergence for degenerate problems

Known fact: Auglag is more robust than interior-point

Property 1: Problem with redundant constraints

With the dual regularization r, the subproblem (NCL_k) automatically satisfy LICQ

Property 2: Infeasible problems

For infeasible problems, NCL converges to a stationary point of the nonlinear least-square problem:

$$\min_{x,r} \frac{\rho}{2}\|r\|^2 \quad \text{subject to} \quad c(x) + r = 0\,,\ x \geq 0$$

Property 3: Problems with complementarity constraints (MPCC)

For MPCC problem, Auglag converges to a strong stationary point (under specific technical assumptions)

AF. Izmailov, MV. Solodov, and EI. Uskov. "Global convergence of augmented Lagrangian methods applied to optimization problems with degenerate constraints, including problems with complementarity constraints." SIAM Journal on Optimization 22 (2012)

Part III: Solving large-scale SCOPF

- How to formulate SCOPF with complementarity constraints?
- How fast can NCL solve the SC-OPF problem?

An application to Corrective Security-Constrained OPF

Suppose we have K potential contingencies, with for each $k = 1, \cdots, K$,

Automatic generation control system (droop control)

$$p_g^k = \min\left(\max\left(p_g^0 + \alpha_g \Delta^k,\ \underline{p}_g\right),\ \overline{p}_g\right)$$

or, equivalently,

$$\begin{aligned}
&\rho_{g,+}^k - \rho_{g,-}^k = p_g^k - (p_g^0 + \alpha_g \Delta) \\
&0 \leq \rho_{g,-}^k \perp \overline{p}_g - p_g^k \geq 0 \\
&0 \leq \rho_{g,+}^k \perp p_g^k - \underline{p}_g \geq 0
\end{aligned}$$

Voltage control (PV/PQ switches)

$$\begin{aligned}
&\nu_+^k - \nu_-^k = v_m^k - v_m^0 \\
&0 \leq \nu_-^k \perp \overline{q}_g - q_g^k \geq 0 \\
&0 \leq \nu_+^k \perp q_g^k - \underline{q}_g \geq 0
\end{aligned}$$

I. Aravena et al. "Recent developments in security-constrained AC optimal power flow: Overview of challenge 1 in the ARPA-E grid optimization competition." Operations research 71 (2023)

Contingency screening

Objective: Detect all the infeasible contingencies

Let x_0 be a base case solution, x_k the variables at contingency k
We abstract the complementarity constraints as:

$$0 \leq \tau_k^L(x_0, x_k) \perp \tau_k^U(x_k) \geq 0$$

Feasibility problem

$$\min_{x_k} \; 0 \quad \text{subject to} \quad \begin{cases} c_k(x_k) = 0 \;, \quad x_k \geq 0 \\ 0 \leq \tau_k^L(x_0, x_k) \perp \tau_k^U(x_k) \geq 0 \end{cases}$$

Using a smooth reformulation of the complementarity cons., NCL solves the nonlinear least-square problem (with MPCC), for $r_k = (r_{c,k}, r_{l,k}, r_{r,k}, r_{s,k})$,

$$\min_{x_k, r_k} \; \frac{1}{2}\|r_k\|^2 \quad \text{subject to} \quad \begin{cases} c_k(x_k) + r_{c,k} = 0 \;, \quad x_k \geq 0 \\ 0 \leq \tau^L(x_0, x_k) + r_{l,k} \\ 0 \leq \tau^R(x_k) + r_{r,k} \\ \tau^L(x_0, x_k)^\top \tau^R(x_k) + r_{s,k} \leq 0 \end{cases}$$

FE. Curtis, DK. Molzahn, S. Tu, A. Wächter, E. Wei, E. Wong.
"A decomposition algorithm with fast identification of critical contingencies for large-scale security-constrained AC-OPF." Operations Research 71 (2023)

SCOPF problem

Objective: adjusting the base case x_0

We keep the K_s most important contingencies in the problem (generally, $K_s \approx 10$)

Corrective SCOPF

$$
\begin{aligned}
\min_{x0, x_1, \cdots, x_{K_s}} \quad & f(x_0) \\
\text{subject to} \quad & c_0(x_0) = 0\,, \quad x_0 \geq 0 \\
& c_k(x_k) = 0\,, \quad x_k \geq 0 \qquad \forall k = 1, \cdots, K_s \\
& 0 \leq \tau_k^L(x_0, x_k) \perp \tau_k^U(x_k) \geq 0 \quad \forall k = 1, \cdots, K_s
\end{aligned}
$$

- Complicated problems with many complementarity constraints
- Fortunately, NCL is also here to help!

(Preliminary) Numerical results on corrective SCOPF

Observations

- Both Knitro and Ipopt don't converge on these instances
- NCL converges to stationary points
- Code runs on the GPU, but is less robust than on the CPU – yet (ill-conditioning in K_{1s})
- If convergence achieved, we observe again a 10x speed-up w.r.t. the CPU

		NCL+K2r+MA27					NCL+K2r+CUDSS					NCL+K1s+CUDSS				
	K	st	obj	it	linsolve	**total**	st	obj	it	linsolve	**total**	st	obj	it	linsolve	**total**
1354pegase	16	1	7.4	282	235.3	**259.7**	-3	7.4	295	30.0	**35.0**	1	7.4	231	17.9	**21.1**
ACTIVSg2000	8	1	122.9	296	543.2	**564.1**	1	122.9	314	29.1	**33.9**	-3	122.9	429	31.7	**37.0**
2869pegase	8	-3	13.4	331	305.0	**340.0**	1	13.4	211	21.5	**26.7**	-3	13.4	244	19.1	**23.2**

Table: st: return status (1 if locally optimal, -3 if step is becoming too small)

Conclusion

Take away

1. We obtain promising (and unexpected) results
   when solving MPCC in the large-scale regime
2. More rigorous analysis is under way
   (we should compare with an exact-penalty based approach)